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CHAOS IN TWO-PARTY DEMOCRACIES

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In two-party democracies (e.g. US and UK) the two parties often alternate at the government almost periodically. Here, we present a very simple continuous-time model with three state variables (social welfare and size of the lobbies associated with the two parties) that explains this tendency to cyclic behavior. However, the analysis of the model shows that when the lobbies are unbalanced, much more complex behaviors, including chaos, can emerge. The bifurcation structure of the system is interesting: it contains a countable number of codimension-2 points (associated with pseudo-equilibria of a particular Filippov system) which are the roots of Arnold tongues delimited by two border collision bifurcations, and in each one of these tongues the sequence of the parties at the government is a particular periodic sequence.

Keywords: Chaos; Arnold tongues; democracy; parties; discontinuous systems; border collision.

1. Introduction

In multi-party democracies a party can remain at the government (alone or in coalition) for a very long time. For example, in Italy where there have always been more than ten parties, the Christian Democracy remained uninterruptedly at the government for 44 years, starting in 1948. By contrast, in two-party democracies like the United Kingdom or the United States, the leading party is often not reelected. For example in the US, democrats (D) and republicans (R) have been alternating at the government since 1945 with President Harry S. Truman (D) in the following way

$$DDRRDDRRDRRRDDRR$$

The sequence is not periodic but there is a frequently recurrent pattern, namely $DDRR$, i.e. each president remains in charge for two terms. This tendency to cycle can only be attributed to endogenous mechanisms, while exogenous random factors like trends in the economy, wars, scandals and appeal of the candidates might at most explain deviations from regular behaviors.

Here, we propose a very simple model that mimics in a rather naïve way the dynamics of the lobbies associated with the parties and the criterion that people follow when they vote once every T years. The analysis of the model shows that when the lobbies have comparable characteristics, the two parties alternate regularly at the government and the welfare varies periodically. But this is not so when there are relevant differences in the lobbies. In particular, the model shows that even in the absence of external factors, welfare can vary chaotically and the result of the elections can follow rather complex patterns. These patterns can be explained through bifurcation analysis, by computing Arnold tongues associated to particular pseudo-equilibria

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of the Filippov system describing the limit case of extremely frequent elections. Moreover, the analysis shows that the mean welfare decreases with respect to T and is, in any case, higher than the welfare that each party could guarantee by remaining permanently at the government. This gives a theoretical support to the idea that two-party democracies are more efficient than multi-party democracies.

Even if the results obtained with our model are sensible and in line with historical observations and beliefs, we like to stress that the model is extremely simplified with respect to reality. Thus, this study should essentially be considered as a provocative intellectual exercise, but can however be a good starting point for further refinements.

2. The Model

The model we propose mimics in a very simplified way the mechanisms present in a real world two-party democracy. Our basic assumption is that there exists a lobby associated to each party, and the individuals belonging to the lobby of the party at the government erode the welfare that the government is able to generate, at a rate proportional to the size of the lobby. The size of a lobby can increase only as long as its party is at the government, and decays otherwise since there is always a small fraction of the individuals who abandon their activity. Finally, a small fraction of the individuals belonging to the lobby of the party that is not at the government defect and switch to the other lobby (the turncoats). Once every T years there are elections and people vote for the party that has the less damaging lobby at the time of the elections. This last assumption, which might look very crude at a first glance, actually explains why in pre-electoral period each party tries to convince the people that the other party is potentially very damaging.

Altogether, the dynamics of the model are captured by three state variables, namely the social welfare (W) and the size of the lobbies (L_D and L_R). In the following we indicate the state of the system by x, i.e.

$$x = \begin{bmatrix} W \\ L_D \\ L_R \end{bmatrix}$$

so that the dynamics of the system will be described by two sets of ODE, namely

$$\dot{x} = f^{(i)}(x) \qquad (1)$$

where $i \in \{D, R\}$ indicates the party at the government. The social welfare W can vary between 0 and 1, where 1 is the highest possible level of welfare, and the sizes of the lobbies are assumed to be reals. The two vector fields $f^{(i)}$ are:

$$f^{(D)}(W, L_D, L_R) = \begin{cases} r(1 - W - a_D L_D)W, \\ (e_D a_D W - d_D)L_D + k_R L_R, \\ (-d_R - k_R)L_R, \end{cases}$$

$$f^{(R)}(W, L_D, L_R) = \begin{cases} r(1 - W - a_R L_r)W, \\ (-d_D - k_D)L_D, \\ (e_R a_R W - d_R)L_R + k_D L_D. \end{cases}$$

Here, r is the intrinsic growth rate of the welfare, which specifies the speed at which the welfare increases in the ideal case of no lobbies; a represents the aggressiveness of a lobby, namely the rate at which the social welfare decreases per unit size of the lobby; e is the recruitment coefficient of a lobby, that is, the proportionality factor between the flow of eroded welfare and the flow of new individuals entering the lobby; d and k are respectively the rate at which individuals abandon the lobbies or defect. In the region S_D (S_R) of the state space where $a_D L_D < a_R L_R$ ($a_D L_D > a_R L_R$) the D-lobby (R-lobby) is less damaging, therefore people will vote for the D-party (R-party) if the state x of the system is in S_D (S_R) at the time of the elections.

In principle, the time T between elections should be a small natural number (for example four years for the US). Nonetheless, the cases $T \to \infty$ and $T = 0$ deserve some attention, because they help understand the properties of the model.

The case $T = \infty$ corresponds to putting one party at the government and leaving it there forever. One can prove that under these conditions there are no limit cycles (or more complex attractors) and only one stable equilibrium, namely

$$\bar{x}^D = \begin{bmatrix} \dfrac{d_D}{e_D a_D} \\ \dfrac{e_D a_D - d_D}{e_D a_D^2} \\ 0 \end{bmatrix}, \quad \bar{x}^R = \begin{bmatrix} \dfrac{d_R}{e_R a_R} \\ 0 \\ \dfrac{e_R a_R - d_R}{e_R a_R^2} \end{bmatrix}$$

if $e_i a_i > d_i$, and

$$\bar{x}^D = \bar{x}^R = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

otherwise.

Notice that, if $e_i a_i > d_i$ for both parties, then $\bar{x}^D \in S_R$ and $\bar{x}^R \in S_D$. Therefore, if T is large but finite, the system can only oscillate roughly between $\bar{x}^D$ and $\bar{x}^R$, remaining for a long time close to these points but never reaching them. In other words, the state of the system must tend to a limit cycle or to a more complex attractor. In this case, the mean value of the welfare $\overline{W}_T$ must tend for $T \to \infty$ to the mean between the values $\overline{W}^D$ and $\overline{W}^R$ that the two parties would obtain if they remained at the government forever, i.e. $\overline{W}_\infty = (1/2)(\overline{W}^D + \overline{W}^R)$.

In the case $T = 0$ elections are held continuously. This has obviously no physical meaning, and too frequent elections would introduce negative effects that are not considered in the model. However, from a purely mathematical point of view, the model becomes a Filippov system [Filippov, 1964, 1988], where the state space is divided in two regions (S_D and S_R) by the discontinuity boundary $a_D L_D = a_R L_R$. Trajectories evolve in accordance with the vector field $f^{(D)}$ in region S_D, and $f^{(R)}$ in region S_R. All trajectories eventually reach the discontinuity boundary and slide on it from there on. The evolution of the state on the discontinuity boundary can be interpreted as a sequence of infinitely short segments of trajectory obeying the vector fields $f^{(D)}$ or $f^{(R)}$ or, equivalently, it can be viewed as a smooth trajectory obeying the so-called Filippov sliding vector field:

$$f^{(S)} = \lambda f^{(D)} + (1-\lambda) f^{(R)}, \qquad (2)$$

where λ is between 0 and 1 and depends on the system state in such a way that the sliding vector field remains tangent to the discontinuity boundary. Moreover, one can prove that the system restricted to the discontinuity boundary has a stable equilibrium x^* (called pseudo-equilibrium) and no other attractor. The pseudo-equilibrium x^* is characterized by $f^{(S)} = 0$, i.e. by

$$\lambda f^{(D)} + (1-\lambda) f^{(R)} = 0,$$

and trivial but cumbersome computations show that the mean value of the welfare at the pseudo-equilibrium is well approximated by $(\overline{W}^D + \overline{W}^R)$ if k_D and k_R are small. This means that if the turncoats are not too many, then $\overline{W}_0 = 2\overline{W}_\infty$, i.e. the average welfare is higher when the elections are extremely frequent than when they are very seldom.

3. Model Behavior

In order to facilitate readers who are not familiar with the analysis of dynamical systems, we discuss in this section the behavior of the model without, for the moment, making reference to bifurcations. This means that we show the result of a huge number of simulations (almost one million for some figures), while we relegate to the next section the analysis and discussion of the complex structure of these figures.

We have already seen that, as long as $e_i a_i > d_i$ for both parties, the system temporarily evolves, during each term, toward $\bar{x}^D$ or $\bar{x}^R$. Therefore, as long as the values of a_D and a_R are sufficiently large, the asymptotic behavior must be periodic, quasi-periodic or chaotic. In order to detect all modes of behavior of the model we have simulated the system for different values of the term length (T) and of the attack rate (a_D) of party D, keeping all other parameters fixed at the values indicated in the caption of Fig. 1. In Fig. 1 the colors indicate the number of terms in a cycle, where blue stands for few terms and red for many terms. Notice that we have stopped counting after 256 elections: thus the dark red regions in the upper part of the figure could correspond to orbits of much higher period or to aperiodic behaviors. In particular, in points ①, ②, ③ and ④ the cycles have 2, 3, 5 and 6 terms respectively, while in point ⑤ the attractor is chaotic. In point ⑥ and in the surrounding region, which is characterized by $e_D a_D < d_D$, party D remains at the government indefinitely. We see that in the central part of the figure, where the two lobbies have similar characteristics, the attractor of type ① is most frequently encountered. In other words, when the two lobbies are similar, the government changes at every election and the welfare varies periodically. By contrast, in the red regions the attractors are more complex, so that one could expect the outcome of the elections and the course of the welfare to be more unpredictable. Actually, this is only partly true, because there are chaotic attractors characterized by strictly periodic sequences of the parties at the government. This becomes clear by looking at Fig. 2. Here the colors do not correspond to the number of terms in a cycle, but to the number of terms in a periodic sequence of the parties at the government. For example, a four-term cycle where the parties alternate at each election ($DRDR$) has a periodic sequence of parties at the government (DR) involving only two terms. The central blue region is much larger than in Fig. 1, meaning that many of the complex attractors that we observe in Fig. 1 are actually very simple if we look only at the outcomes

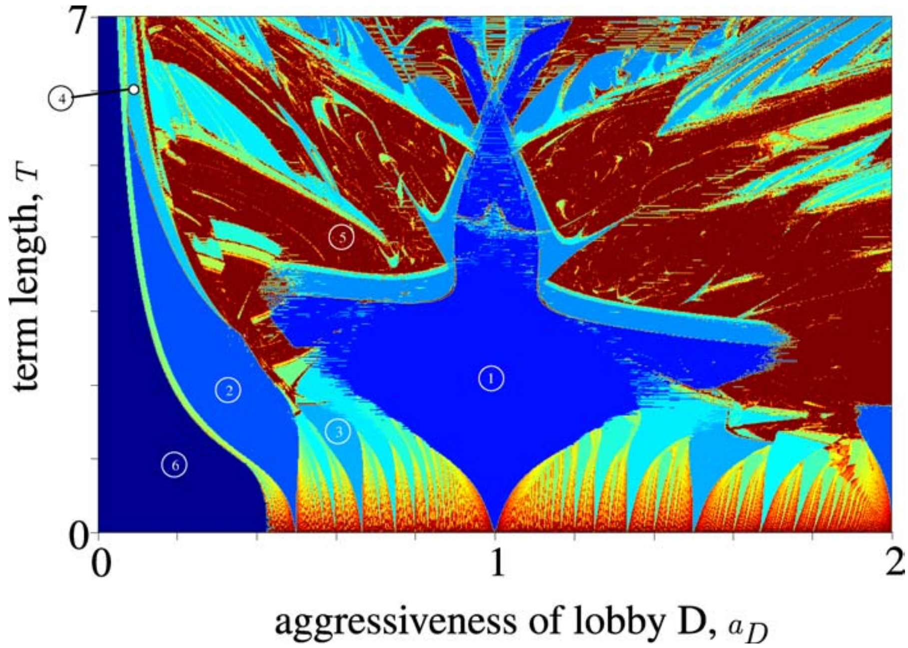


Fig. 1. The colors, from blue to red, correspond to the increasing number of terms in a cycle. The value of the parameters are: $a_R = 1$, $r = 0.2$, $e_D = e_R = 6$, $d_D = d_R = 1.8$, $k_D = k_R = 0.06$.

of the elections. On the other hand, if the differences between the two lobbies are more relevant, then also the outcomes of the elections can become more complex.

We can obtain some more information by observing how the mean value $\overline{W}_T$ of the welfare changes with respect to T and a_D. We already know from the previous section that $\overline{W}_0 > \overline{W}_\infty$, but Fig. 3 shows that the average welfare $\overline{W}_T$ actually decreases with respect to T, thus suggesting that shorter terms are more effective. Another (more intuitive) outcome of Fig. 3 is that the mean welfare decreases with respect to the aggressiveness.

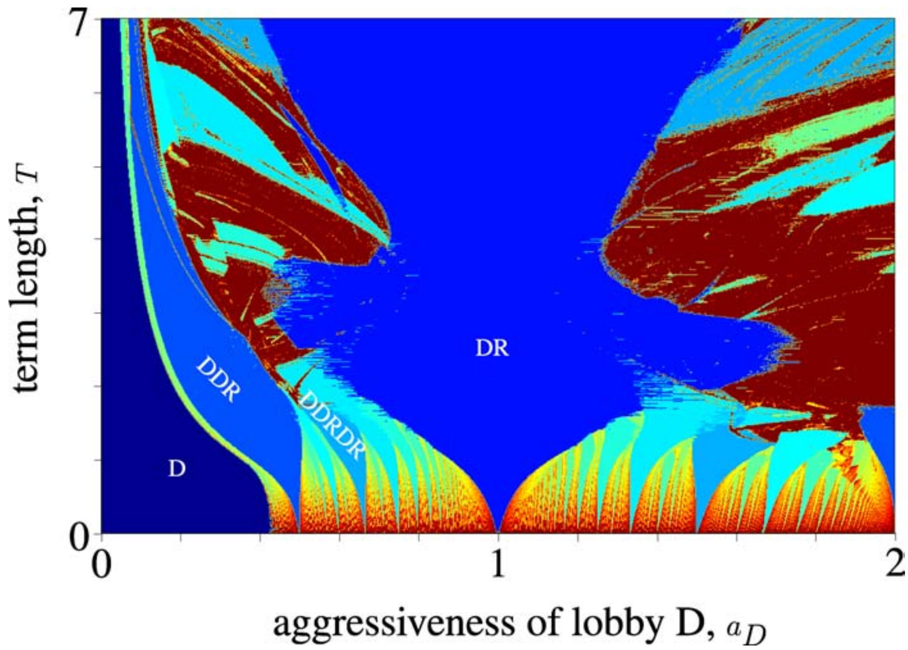


Fig. 2. The colors, from blue to red, correspond to the increasing number of terms in a periodic sequence of parties at the government. In some of the main regions in the figure these sequences are explicitly pointed out.

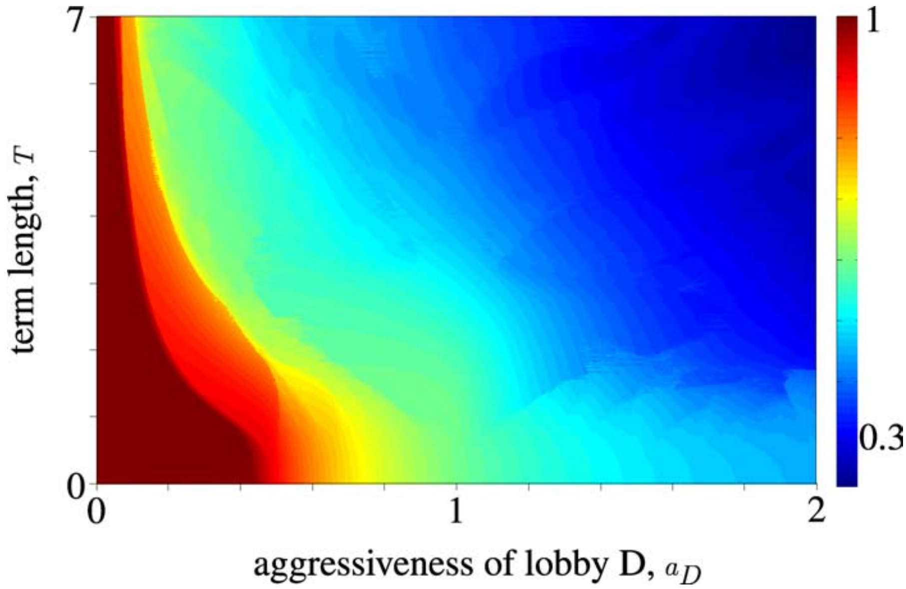


Fig. 3. The average value $\overline{W}_T$ of the welfare for different values of the aggressiveness a_D and of the term length T. Notice that the welfare decreases with respect to T and a_D.

As the reader has certainly noticed, Figs. 1 and 2 display an obvious structure, that is studied in the next section in terms of bifurcations.

4. Bifurcation Analysis

Model (1) is a discontinuous system with periodic and chaotic attractors. Varying its parameters one can, therefore, expect to find flip and tangent bifurcations of limit cycles, as well as other nonstandard bifurcations that are peculiar to discontinuous systems. In particular, a nonstandard bifurcation occurs every time a vertex of a cycle (i.e. a point of a cycle corresponding to an election) touches the discontinuity boundary. This bifurcation is similar to the border collision bifurcation (also known as c-bifurcation) that can be found in piecewise-smooth maps [Feigin, 1970, 1974, 1978; Nusse & Yorke, 1992, 1995; Nusse *et al.*, 1994]. This is not surprising, as our system can be seen as a piecewise-smooth map, where the next vertex is obtained from the present vertex by integrating the appropriate vector field ($f^{(D)}$ or $f^{(R)}$) for a time T. Since this map is discontinuous across the boundary, after the border collision the cycle generically disappears and the system settles on another attractor.

Given the complex structure of Fig. 1, a detailed bifurcation diagram would be difficult to read and of little help. For this reason, we focus only on the few bifurcations that surround the larger regions in Fig. 1. In particular, we report in Fig. 4 the bifurcations of stable cycles, found through continuation, that surround the regions around point ① and ② in Fig. 1. The upper boundaries of these regions consist of flip (F) and tangent (TC) bifurcations of limit cycles. The flips are connected to border collisions (BC) at codimension-2 bifurcation points. A complete unfolding of this codimension-2 points would show that one flip and two border collisions of stable cycles branch out from these points, since in a neighborhood of the codimension-2 point both stable cycles involved in the flip must undergo a border collision. However, in order to keep the diagram simple, the border collisions of the cycle of longer period have not been drawn in Fig. 4. Moving further down in the

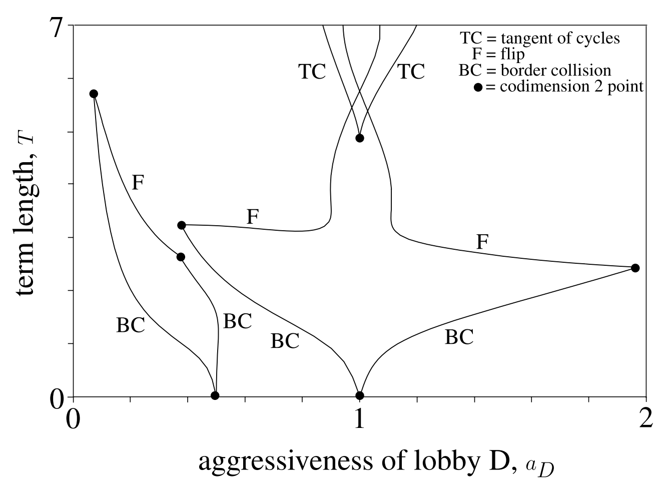
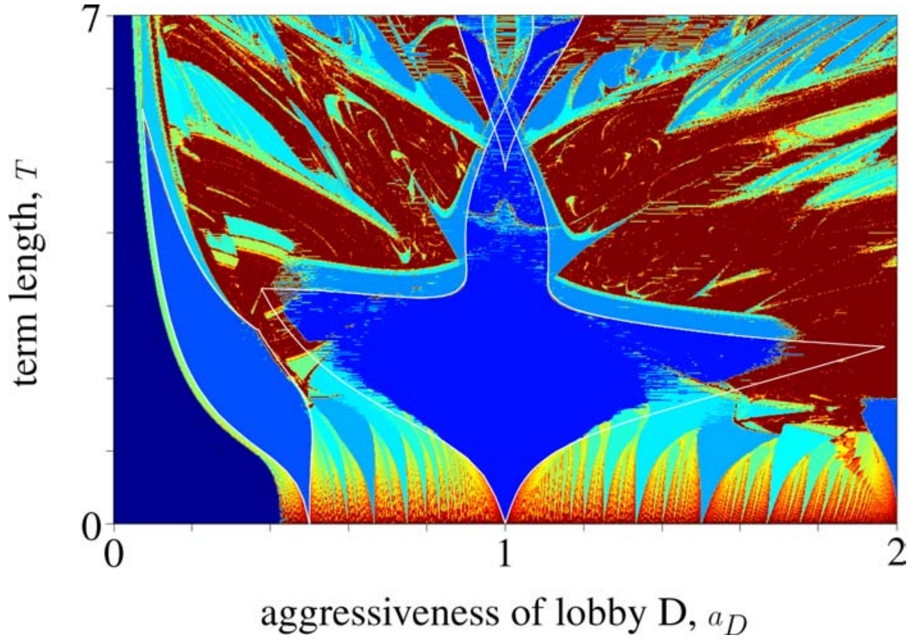
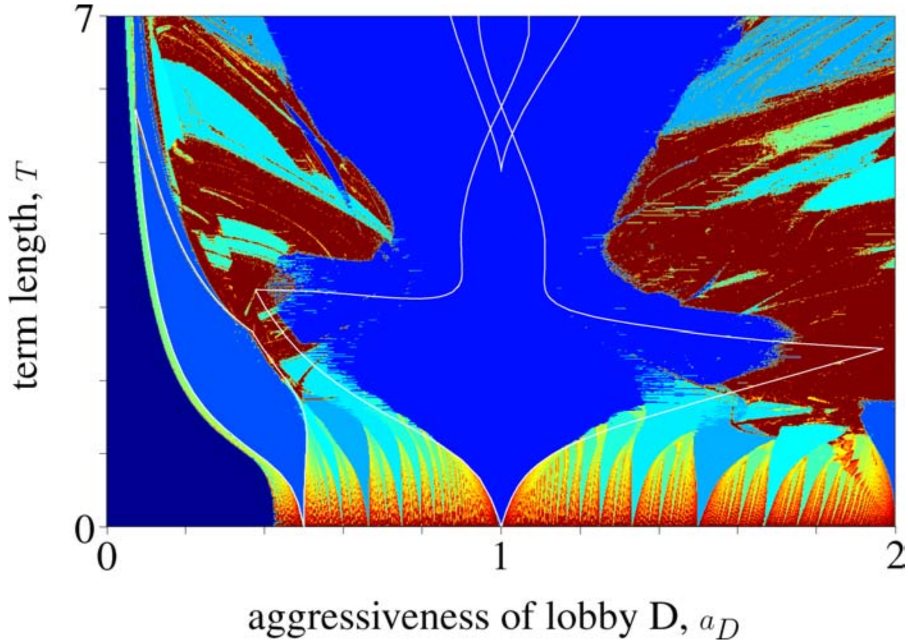


Fig. 4. The main bifurcation curves of model (1), obtained through continuation.



(a)



(b)

Fig. 5. The main bifurcation curves superposed to Figs. 1 and 2.

diagram, couples of border collisions form Arnold tongues, rooted at codimension-2 points on the $T = 0$ axis, forming a structure similar to the one observed in particular piecewise-smooth maps [Feely, 1991; Jain & Banerjee, 2003; Avrutin *et al.*, 2006; Hogan *et al.*, 2007]. Looking at Fig. 1, we see that the same structure is repeated a large number of times, i.e. the $T = 0$ axis is dense with such codimension-2 points. This fact can be easily explained considering the behavior of the system extremely close to the horizontal axis. As we have seen, when $T = 0$ the model becomes a Filippov system, whose only attractor is a pseudo-equilibrium of the sliding vector field (2). If T is small, we can then expect a small cycle to exist instead of the pseudoequilibrium. Increasing the aggressiveness a_D, this small cycle hits the discontinuity boundary with one of its vertices, and the same occurs (with another vertex) if the aggressiveness is decreased. This explains the presence of the two border collision curves delimiting the Arnold tongue. A more careful analysis can take us even further, and allow the prediction of the exact position of the Arnold tongues and the structure of the corresponding cycles. In fact, the ratio of the number of D-terms to the total number of terms in the cycle must approach λ in (2) as $T \to 0$, so that if λ is rational, say p/q, the cycle must be composed of p D-terms and $q - p$ R-terms. The sequence of D-terms and R-terms is then uniquely determined following the election rules. This means that an Arnold tongue is rooted on the $T = 0$ axis at each point where λ (evaluated at the pseudo-equilibrium) is rational. The general formula for λ in terms of the other parameters is quite complicated, but assuming, as we did in our simulations, that $a_R = 1$, $e_R = e_D$, $k_R = k_D$, $d_R = d_D$, we obtain that the dependence of λ on a_D assumes the beautifully simple form

$$\lambda = \frac{1}{1 + a_D}.$$

It is now easy to see that, in Fig. 2, the tongue rooted at $a_D = 1$ must contain a cycle of two terms (DR), the one rooted at $a_D = 0.5$ a cycle of three terms (DDR), and the one rooted at $a_D = 2/3$ a cycle of five terms ($DDRDR$).

This concludes our short analysis of the bifurcations that support the structure of Figs. 1 and 2. For more clarity, we summarize our results in Fig. 5, where the bifurcation lines analyzed in this section are superposed to Figs. 1 and 2.

5. Conclusions

In this paper, we present a model of a two-party democracy, based on some simplistic but reasonable assumptions. We use a discontinuous system, with three state variables and two sets of differential equations, each one describing the effect of one party at the government. The dynamics we observe through simulations show a complex interplay of different periodic and chaotic attractors, organized by a set of flip, tangent of cycles and border collision bifurcations. A more accurate analysis of these bifurcations, done through continuation, allows to explain most of the observed structures.

The work we have done represents a first step in the application of dynamical systems in this field, so that further extensions are not only possible but advisable. In particular, obvious extensions would include considering different criteria for the choice of the party at the time of elections, more accurate models for the dynamics of the welfare, and entrainment effects that could justify the frequent re-election of the same party for two consecutive terms, as in the US.

As far as our study is concerned, we can state two main results. First, the model supports the idea that, independently of the complex influences of external factors, long-term predictions of the electoral outcomes in the real world can be very difficult, if not impossible, due to the intrinsically chaotic dynamics of the political system. Second, a two-party system is better than a multi-party system because, ensuring a more frequent change of the party at the government, it can achieve a higher level of welfare. Once again though, we must stress that this model is extremely simple and has no ambition of giving a complete or even faithful description of the phenomena it models.

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